Cart Pole Control

June 3, 2024

All the parameters used in this document have already been introduced in the corresponding blog post, except T_x and T_y which are respectively the reaction force respectively on the x and y axis.

1 System dynamics

Newton's Second Law on the **pole**:

$$m_p \ddot{x}_G = T_x \tag{1}$$

$$m_p \ddot{y}_G = T_y - m_p g \tag{2}$$

Angular moment theorem on the **pole**:

$$I\ddot{\theta} = l - (T_x \cos\theta + T_y \sin\theta) \tag{3}$$

Newton's Second Law on the **cart**:

$$m_c \ddot{x} = F - T_x \tag{4}$$

1.1 Kinematics relationships

Coordinates of the gravity center of the pole are:

- $x_G = x - l\sin\theta$
- $y_G = l\cos\theta$

We differentiate them to get velocities:

- $\dot{x}_G = \dot{x} - l\dot{\theta}\cos\theta$
- $\dot{y}_G = -l\dot{\theta}\sin\theta$

We differentiate them again to get accelerations:

$$\ddot{x}_G = \ddot{x} - l\ddot{\theta}\cos\theta + l\dot{\theta}^2\sin\theta \tag{5}$$

$$\ddot{y}_G = -l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta \tag{6}$$

Injecting (5) into (1) and (6) into (2), we have:

$$T_x = m_p(\ddot{x} - l\ddot{\theta}\cos\theta + l\dot{\theta}^2\sin\theta) \tag{7}$$

$$T_y = m_p(g - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta) \tag{8}$$

Then we replace T_x and T_y in (3):

$$\ddot{\theta} = \frac{lm_p}{I + l^2 m_p}(\ddot{x}\cos\theta + g\sin\theta) \tag{9}$$

As the solid is a pole, we have: $I = \frac{m_p l^2}{3}$. Thus:

$$\boxed{\ddot{\theta} = \frac{3}{4l}(\ddot{x}\cos\theta + g\sin\theta)} \tag{10}$$

On the other side, injecting (7) into (4), we have:

$$m_c\ddot{x} = F - m_p(\ddot{x} - l\ddot{\theta}\cos\theta + l\dot{\theta}^2\sin\theta) \tag{11}$$

$$\boxed{\ddot{x} = \frac{1}{m_c + m_p}\left(F + m_p l\ddot{\theta}\cos\theta - m_p l\dot{\theta}^2\sin\theta\right)} \tag{12}$$